Penalty Function Solution of Steady-State Navier-Stokes Equations

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Introduction

IN the numerical solution of the Navier-Stokes equations for steady, incompressible viscous flow using the finite-element method, the use of primitive fluid variables in two dimensions offers advantages which have led to an increasing interest in their use, especially when extensions to three-dimensional flows are in mind.[1]

The main problem faced when primitive variables are considered arise from imposing the incompressibility constraint. This usually has been achieved by weighting the continuity equation using the interpolation functions for the pressure in a Galerkin formulation,[2] and, in some cases, through the use of interpolation functions for the velocity field that satisfy the continuity equation a priori, element-wise.[3] A third possibility is the addition of a penalized term to the Galerkin form of the momentum equations with the advantage that the pressure is eliminated as a dependent variable. The latter approach has been called the penalty function finite-element method.[4-6] In this Note, conclusions obtained from extensive quantitative evaluation using three types of rectangular elements are discussed, and the advantages of using biquadratic interpolation functions are pointed out.

We consider the solution of the two-dimensional, Navier-Stokes equations for an incompressible fluid with constant physical properties

$$\rho u_j \frac{\partial u_i}{\partial x_j} = \rho F_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

where ρ is the density, F_i applied body forces, and μ the viscosity. p denotes the pressure and u_i the velocity components in the x_i Cartesian coordinate direction. Equations (1) and (2) are assumed to hold in a region $\Omega \subset R^2$ with boundary Γ and boundary conditions

$$\boldsymbol{u} = \boldsymbol{u}^\circ \qquad \text{in } \Gamma_1 \tag{3}$$

$$\boldsymbol{t} = \boldsymbol{t}^\circ \qquad \text{in } \Gamma_2 = \Gamma - \Gamma_1 \tag{4}$$

Here u° (the vector with components u_i°) is a prescribed velocity on a portion Γ_1 of the boundary and t° denotes a prescribed boundary traction on the rest of the boundary Γ_2, with $\Gamma_1 \cap \Gamma_2 = \Phi$, the empty set.

Received June 27, 1978; revision received Feb. 5, 1979.

Index categories: Computational Methods; Analytical and Numerical Methods.

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Penalty Function Finite-Element Method

A velocity field $\boldsymbol{u}$ which is a solution of Eqs. (1) and (2) satisfies[7]

$$A(u,v) \equiv \int_\Omega v_i \rho \left(u_j \frac{\partial u_i}{\partial x_j} - F_i\right) + \mu \frac{\partial v_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \mathrm{d}\Omega = 0 \tag{5}$$

for all $\boldsymbol{v}$ in the space $H^1(\Omega)$ of vector functions whose components are square-integrable, together with its first partial derivatives and, furthermore, satisfy Eq. (2).

In a finite-element approximation, continuity can be satisfied ony globally. Locally, it is approximately satisfied unless the interpolation functions are chosen to do so in some sense.[3] If the standard rectangular Lagrangian or serendipity interpolation functions[8] are used, an approximation to the velocity field can be computed by augmenting Eq. (5) with a penalty term

$$P_\lambda(u,v) = \lambda \int_\Omega \frac{\partial v_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \mathrm{d}\Omega \tag{6}$$

where λ, the penalty parameter, is a large number. The problem is now to find $\boldsymbol{u}$ such that

$$A(u,v) + P_\lambda(u,v) = 0 \tag{7}$$

for all $\boldsymbol{v}$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are now the appropriate finite-dimensional space determined by the discretization. This leads to the solution of an algebraic system of the form

$$[\boldsymbol{K}_1(\boldsymbol{u}) + \lambda \boldsymbol{K}_2]\boldsymbol{u} = \boldsymbol{F} \tag{8}$$

which is typically solved using Newton's method to treat the nonlinearity of $\boldsymbol{K}_1$.

For the linear problem, a convergence proof and error estimates when $\lambda \to \infty$ have been given.[6] However, in finite-element approximations, the fact that the trial and test functions are in the same space overconstrains the problem; this is overcome by the use of reduced integration[8] in the evaluation of the matrix $\boldsymbol{K}_2$ and is equivalent to reducing the number of linearly independent constraints imposed in the resuting system of linear equations. A detailed bibliography on the present method is given in Ref. 9.

The question remains on how to determine approximations of the pressure which is not obtained from solving Eq. 8. Clearly, there are several ways to do this, one direct way being given in Ref. 5. This question is not addressed here.

Use of Biquadratic Elements

In the implementations of the penalty method reported in the literature,[5,10,11] only the simplest four-noded bilinear elements have been used, although quadratic and cubic elements, both of Lagrangian and serendipity[8] type, can be used as well. Experiments with three of these, namely the four-noded bilinear element, the eight-noded quadratic (serendipity), and nine-noded biquadratic (Lagrangian) elements, have been performed through a series of standard problems and show some clear advantages in the use of biquadratic elements, as well as an excellent approximate satisfaction of the incompressibility constraint by the flowfields obtained with the penalty finite-element method.

The eight-noded element shows no advantage over the nine-noded biquadratic element, except for the fact that it is the only one of the three that can be implemented using reduced integration in both matrices $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ without introducing hourglass[8] modes.

Bilinear and biquadratic elements have been used with exactly the same number of nodes (one biquadratic element covers four bilinear ones). Some of their differences can be

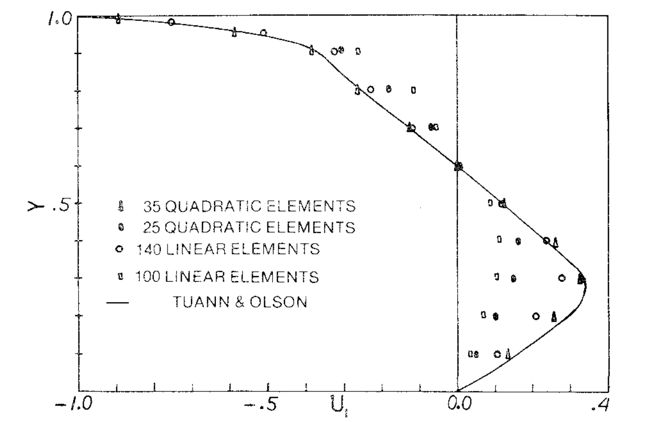

Fig. 1 Horizontal velocity profiles along $x_1 = 0.5$ for Reynolds number 400; bilinear and biquadratic elements.

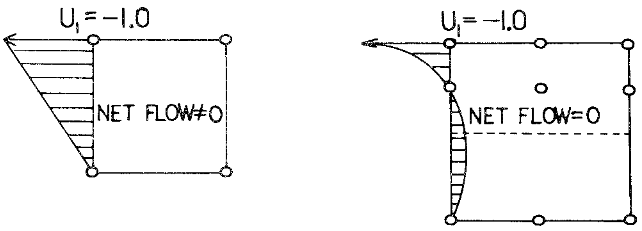

Fig. 2 Mass exchange in a bilinear element (left) at the top left corner (0,1). No net flow n biquadratic elements (right) achieved by shifting of the midnodes.

shown through the driven cavity flow problem. This is solved over the domain $\Omega = [0,1] \times [0,1]$ with boundary conditions $u_1 = u_2 = 0$ on Γ, except along $x_2 = 1$, $0 \leq x_1 \leq 1$, where $u_1 = -1$. Figure 1 shows the horizontal velocity profiles u_1 along the midplane $x_1 = 0.5$ for Reynolds number 400 obtained with bilinear and biquadratic elements in an 11×11 regular mesh, compared with that given by Tuann and Olson.[12] Although biquadratic elements are more accurate than bilinear, both solutions are poor. This is due to the singularities in the horizontal velocity component u_1 at the top corners, which allow a mass exchange to take place within Ω, with the consequent mass imbalance across any internal line $x_1 = a$. This flux is given by $\Delta x_2/2$, where x_2 is the (nondimensional) length of the corner elements in the x_2 direction. The situation is illustrated in Fig. 2. It is clear that the only possible way‡ to avoid this with bilinear elements is the use of a top row of elements thin enough to make the amount of mass exchanged not significant. On the other hand, biquadratic elements provide us with the possibility of shifting the midnodes to balance the flow across the element and provide accurate solutions even for coarse meshes. In Fig. 1 we show two more solutions obtained in a 15×11 mesh obtained by addition of nodes at $x_2 = 0.85$, 0.93, 0.96, and 0.98. The better accuracy of biquadratic elements is significant. Solutions with and without shifting the midnodes at the top were computed, but only the first is shown for clarity.

Conclusions

The use of biquadratic elements with the penalty method not only provides better accuracy but opens a wider range of possibilities to attack pathological situations such as arise in the driven cavity flow. The penalty method provides excellent mass conservation as evidenced by numerical experiments where it can be controlled to within the machine capacity. Further evidence is given by the fact that stream function values calculated by integration of computed velocity field are virtually path independent.[13]

‡The singularity can be eliminated by setting $u_1 = 0$ at the corners, but this fails if bilinear elements are used. Corner elements have only one free node to balance the flow and force the appearance of a checkerboard mode. Quadratic elements can, however, be used.

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Singularities in Unsteady Boundary Layers

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I. Introduction

SINGULARITIES of various kinds are known, or feared, to occur in solutions of the thin-shear-layer ("boundary-layer") equations. The example most commonly quoted is the Goldstein[1] "square root" singularity, in which surface shear

Received Aug. 29, 1978; revision received Feb. 12, 1979.

Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Nonsteady Aerodynamics.

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